Graph $f(x) = \dfrac{x}{(4-x)^3} + 2$ using the process shown in lecture and in the website handout. **SCORE: _____ / 40 PTS**

The first and second derivatives are $f'(x) = (4-x)^{-3} + 3x(4-x)^{-4}$ and $f''(x) = 6(4-x)^{-4} + 12x(4-x)^{-5}$.

<u>Do NOT find x–intercepts.</u>

<u>Complete the table below, after showing relevant work (except for entries marked ★).</u>

<u>You will NOT receive credit for the entries in the table if the relevant work is missing.</u>

★ Domain	★ Discontinuities	y – intercepts **ONLY**	One sided limits at each discontinuity (write using proper limit notation)	
$x \neq 4$	$x = 4$	$(0,2)$	$\lim_{x\to 4^+} f(x) = -\infty$ $\lim_{x\to 4^-} f(x) = \infty$	
Horizontal Asymptotes	Intervals of Increase	Intervals of Decrease	Intervals of Upward Concavity	Intervals of Downward Concavity
$y = 2$	$(-2,4)$, $(4,\infty)$	$(-\infty,-2)$	$(-4,4)$	$(-\infty,-4)$ $(4,\infty)$
Vertical Tangent Lines	Horizontal Tangent Lines	Local Maxima	Local Minima	Inflection Points
NONE	$x=-2$	NONE	$(-2, 1\frac{107}{108})$	$(-4, 1\frac{255}{256})$

$f(0) = \frac{0}{4^3} + 2 = 2$

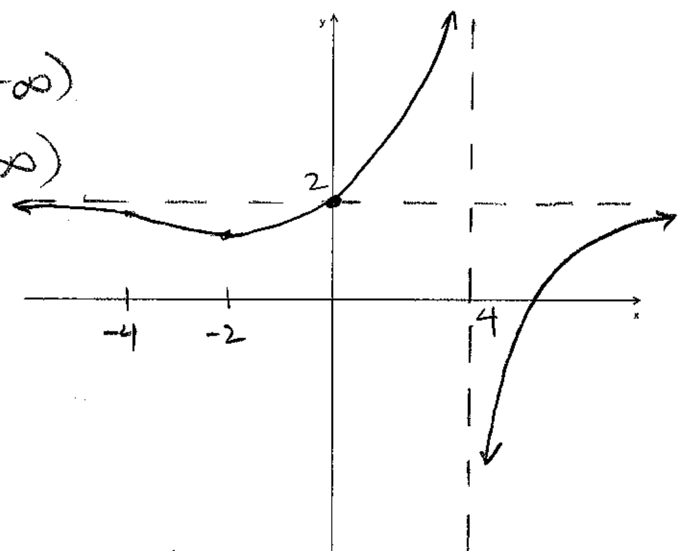
$\lim_{x\to 4^+} \left(\frac{x}{(4-x)^3} + 2\right) = -\infty \quad \left(\frac{4}{0^-} + 2 \to -\infty + 2 \to -\infty\right)$

$\lim_{x\to 4^-} \left(\frac{x}{(4-x)^3} + 2\right) = \infty \quad \left(\frac{4}{0^+} + 2 \to \infty + 2 \to \infty\right)$

$\lim_{x\to \pm\infty} \left(\frac{x}{(4-x)^3} + 2\right) = \left(\lim_{x\to\pm\infty} \frac{1}{-3(4-x)^2}\right) + 2$

$\frac{\infty}{\pm\infty} + 2$

$= 0 + 2 = 2$



$f'(x) = (4-x)^{-4}(4-x+3x)$

$= (4-x)^{-4}(2x+4)$

$= 2(4-x)^{-4}(x+2)$ UNDEFINED @ $x=4 \notin$ DOMAIN

$= 0$ @ $x=-2$

$f''(x) = 6(4-x)^{-5}(4-x+2x)$

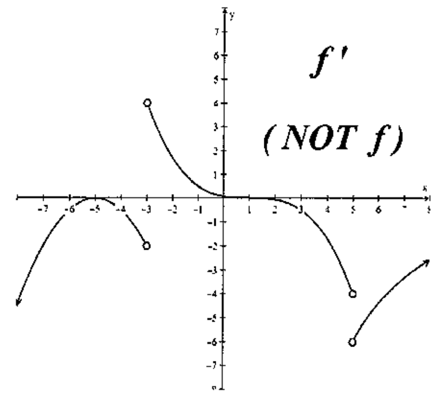
$= 6(4-x)^{-5}(x+4)$ UNDEFINED @ $x=4 \notin$ DOMAIN

$= 0$ @ $x=-4$

		-4		-2		4	
f'	$-$		$-$	LOCAL MIN	$+$		$+$
f''	$-$	IP	$+$		$+$		$-$

$f(x)$ is a continuous function whose derivative $f'(x)$ is shown on the right. SCORE: _____ / 24 PTS

The following questions are about the function f, **NOT THE FUNCTION f'**.



[a] Find the $x-$coordinates of all inflection points of f.
Justify your answer very briefly.

f' CHANGES FROM INCR TO DECR: X = -5
DECR INCR X = 5

[b] Find the intervals over which f is increasing.
Justify your answer very briefly.

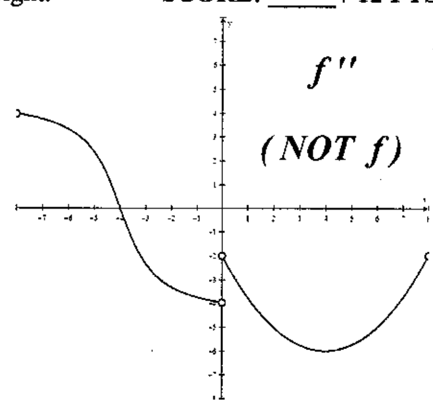
$f' > 0 : (-3, 1)$

[c] Find all critical numbers of f, and state what the First Derivative Test tells you about each one.
Justify your answer very briefly.

$f' = 0$: -5 f' CHANGES FROM $-$ TO $-$ → NOT EXTREMA
1 $+$ $-$ → LOCAL MAX
f' DNE: -3 $-$ $+$ → LOCAL MIN
5 $-$ $-$ → NOT EXTREMA

$f(x)$ is a continuous and differentiable function whose second derivative $f''(x)$ is shown on the right. SCORE: _____ / 12 PTS

The following questions are about the function f, **NOT THE FUNCTION f''**.



[a] If $f'(-6) = 0$,
what does the Second Derivative Test tell you about the point $(-6, f(-6))$?
Justify your answer very briefly.

$f''(-6) > 0$ → LOCAL MIN

[b] Find the intervals over which f is concave up.
Justify your answer very briefly.

$f'' > 0 : (-8, -4)$

Find $\int \frac{(3x^4-2)^2}{5x^5}\,dx$.

SCORE: _____ / 15 PTS

$= \int \frac{9x^8-12x^4+4}{5x^5}\,dx$

$= \int \left(\frac{9}{5}x^3 - \frac{12}{5}x^{-1} + \frac{4}{5}x^{-5}\right)dx$

$= \frac{9}{5}\frac{1}{4}x^4 - \frac{12}{5}\ln|x| + \frac{4}{5}\left(-\frac{1}{4}\right)x^{-4} + C$

$= \frac{9}{20}x^4 - \frac{12}{5}\ln|x| - \frac{1}{5}x^{-4} + C$

Find $\lim\limits_{x\to 0^+} \sqrt[3]{x}\ln x$. $0\cdot -\infty$

SCORE: _____ / 15 PTS

$= \lim\limits_{x\to 0^+} \frac{\ln x}{x^{-\frac{1}{3}}}$ $\quad \frac{-\infty}{\infty}$

$= \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{3}x^{-\frac{4}{3}}}$

$= \lim\limits_{x\to 0^+} -3x^{\frac{1}{3}} = 0$

Does Rolle's Theorem apply to the function $f(x)=\sqrt[3]{x^2-8x+15}$ on the interval $[1, 7]$?

SCORE: _____ / 15 PTS

(That is, are all conditions of Rolle's Theorem true for $f(x)=\sqrt[3]{x^2-8x+15}$ on the interval $[1, 7]$?)
If yes, find the value of c guaranteed by Rolle's Theorem. If no, explain why not.

$f'(x) = \frac{1}{3}(x^2-8x+15)^{-\frac{2}{3}}(2x-8)$

IS UNDEFINED @ $x^2-8x+15=0$

$(x-3)(x-5)=0$

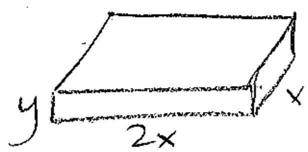
$x=3,5 \in [1,7]$

f IS NOT DIFFERENTIABLE ON $[1,7]$
SO ROLLE'S THEOREM DOES NOT APPLY

300 square inches of material is available to make a box with a rectangular base. The width of the base must be twice its length. The box must include a top. **SCORE: _____ / 30 PTS**

<u>You must use calculus to solve the following problems.</u>

[a] What is the largest possible volume of the box ?



$x \geq 0$ AND $4x^2 \leq 300$

$x^2 \leq 75$

$x \leq 5\sqrt{3}$

MAXIMIZE $V = 2x^2y$

$4x^2 + 6xy = 300$

$2x^2 + 3xy = 150$

$y = \dfrac{150 - 2x^2}{3x}$

$V = 2x^2\left(\dfrac{150-2x^2}{3\cancel{x}}\right) = \dfrac{300x - 4x^3}{3}$ ON $x \in [0, 5\sqrt{3}]$

$V' = \dfrac{300 - 12x^2}{3}$ IS DEFINED ON $[0, 5\sqrt{3}]$

$= 0$ @ $300 = 12x^2$

$25 = x^2$

$x = 5$

x	V
0	0
5	$\dfrac{1500-500}{3} = \dfrac{1000}{3}$
$5\sqrt{3}$	0

THE LARGEST VOLUME IS $\frac{1000}{3}$ CUBIC INCHES

[b] Find the dimensions of the box which give the largest possible volume.

BASE: $5'' \times 10''$

HEIGHT: $\frac{20}{3}''$